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Summability Fields with a Tauberian Transition Property

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1. INTRODUCTION

G. G. Lorentz [1] has developed a general theory of Tauberian theorems, showing the connection between Tauberian gap conditions and Tauberian order conditions. M. Stieglitz [4] continued these investigations. In the recent paper [3] the theory was used to draw a line between Tauberian o- and O-conditions. A variant of the main result there [3, Theorem 2] is Theorem 1.

THEOREM 1. *Let A be a regular matrix with*

$$\liminf_{k\to\infty} \sup_{n=0,1,\ldots} |a_{nk}| = 0. \tag{1}$$

Then there exist constants $d_m > 0$ such that

$$u_m = o(d_m), \qquad \textit{but not} \qquad u_m = O(d_m)$$

is a Tauberian condition for the method A.

Here the method A is based on the sequence-to-sequence transform given by the matrix A with elements a_{nk} ($n, k = 0, 1,\ldots$), and the u_m are the series terms (differences) of the sequence $\mathbf{s} = \{s_k\}$:

$$s_k = \sum_{m=0}^{k} u_m \qquad (k = 0, 1,\ldots). \tag{2}$$

All numbers (e.g., the elements of matrices and of sequences) are complex numbers if nothing is said to the contrary. The reader can find the basic notations of summability in [8].

Condition (1) is not superfluous, as shown by the trivial example A = identity. The question remained open whether a corresponding example with A strictly stronger than convergence could be found. The answer is positive, as shown by Theorem 2, whose proof (together with that of Theorem 8) is the purpose of the present note.

THEOREM 2. *There exists a regular row-finite matrix A which sums certain divergent sequences and which has the following transition property: If the constants $d_m \geqslant 0$ are such that $u_m = o(d_m)$ is a Tauberian condition for A, then also $u_m = O(d_m)$ is a Tauberian condition for A.*

The meaning of o and O here is that there exists a sequence $\{h_m\}$ such that $u_m = h_m d_m$ $(m = 0, 1, \ldots)$ and $h_m = o(1)$ or $= O(1)$, respectively, as $m \to \infty$.

Theorem 2 says in other words that there exists a nontrivial matrix A for which the transition from a Tauberian o-condition to the corresponding O-condition is always possible. The proof consists of two parts. First (Section 2, Theorem 3) we exhibit a sequence space (or rather a class of sequence spaces) with the desired Tauberian property. Then (Section 3, Theorem 8) we show that this space is the convergence domain of a matrix. The approach is based on "Einfolgenverfahren" (cf. [8, p. 48]) and their generalizations (cf. [5, 2]). The construction of such matrices is interesting also from the point of view of general summability theory.

2. A SEQUENCE SPACE

The space which we have in mind will contain (c), the set of convergent sequences, and certain sequences $\mathbf{s}^j = \{s_k^{(j)}\}$ $(j = 0, 1, \ldots)$ with

$$s_k^{(j)} = 0 \qquad \text{for} \quad k = 0, \ldots, q(j) - 1, \tag{3}$$

where the indices $q(j)$ are such that

$$0 \leqslant q(0) < q(1) < \cdots. \tag{4}$$

For the present purpose the $q(j)$ are otherwise arbitrary (one could even relax (4)); later more requirements will be imposed on them recursively. We denote our space by $\mathfrak{S} = \mathfrak{S}(\mathbf{s}^0, \mathbf{s}^1, \ldots)$ and give its definition: $\mathfrak{S}$ consists of all sequences $\mathbf{s} = \{s_k\}$ of the form

$$\mathbf{s} = \tilde{\mathbf{s}} + \sum_{j=0}^{\infty} \lambda_j \mathbf{s}^j, \tag{5}$$

where $\bar{\mathbf{s}} \in (c)$ and $\lambda_j \to 0$. In the latter series of sequences we use coordinatewise convergence; this convergence is assured (even for arbitrary λ_j) because of (3), (4). The main idea is now to use sequences $\mathbf{s}^j$ where $\mathbf{s}^1$ changes less rapidly than $\mathbf{s}^0$, $\mathbf{s}^2$ less rapidly than $\mathbf{s}^1$, and so on. More precisely we demand (for the corresponding series terms, cf. (2); $u_m^{(j)} = u_m(j)$):

$$|u_m(j)| \to \infty, \qquad (m \to \infty; j = 0, 1,...), \tag{6}$$

$$|u_m(j+1)| \leqslant \tfrac{1}{2} |u_m(j)|, \qquad (m = 0, 1,...; j = 0, 1,...), \tag{7}$$

$$u_m(j+1)/u_m(j) \to 0, \qquad (m \to \infty; j = 0, 1,...). \tag{8}$$

For each j the quotient in (8) is defined for all large m.

We provide an example of a set of sequences $\mathbf{s}^j$ fulfilling all the mentioned conditions. Putting

$$w_k^{(j)} = \sum_{m=0}^{k} v_m^{(j)} = \frac{1}{2^j} (k!)^{1/(j+1)}, \tag{9}$$

we check easily that the series terms $v_m^{(j)}$ of the sequences $\mathbf{w}^j$ are of type (6), (7) and (8). Now, given any indices obeying (4) we put

$$u_m^{(j)} = \begin{cases} 0 & \text{for} \quad m = 0,..., q(j) - 1 \\ v_m^{(j)} & \text{for} \quad m = q(j), q(j) + 1,...; \end{cases} \tag{10}$$

then (6), (7), (8), and (3) are fulfilled.

The transformation (10) of the sequences $\mathbf{w}^j$ of our example into corresponding sequences $\mathbf{s}^j$ can be accomplished in the same manner for any given set $\{\mathbf{w}^j\}$. This will be used in Section 3. We describe the transformation (10) also by the formula

$$\mathbf{s}^j = \mathbf{w}^j - \{w_0^{(j)},..., w_{q-2}^{(j)}, w_{q-1}^{(j)}, w_{q-1}^{(j)},...\}, \qquad q = q(j). \tag{11}$$

Now we state:

THEOREM 3. *If the* $\mathbf{s}^j$ *fulfill the conditions* (6), (7), (8), *and* (3), (4), *then the space* $\mathfrak{S} = \mathfrak{S}(\mathbf{s}^0, \mathbf{s}^1,...)$ (*cf.* (5)) *has the following property*: *If the constants* $d_m \geqslant 0$ *are such that any sequence* $\mathbf{s} \in \mathfrak{S}$ *with* $u_m = o(d_m)$ *is convergent, then also any sequence* $\mathbf{s} \in \mathfrak{S}$ *with* $u_m = O(d_m)$ *is convergent.*

In other words: If $u_m = o(d_m)$ is a Tauberian condition for $\mathfrak{S}$, then also $u_m = O(d_m)$ is a Tauberian condition for $\mathfrak{S}$. The theorem remains true if condition $\lambda_j \to 0$ in (5) is replaced by $\lambda_j = O(1)$ (which gives a larger $\mathfrak{S}$). Of course it remains true for any subset of $\mathfrak{S}$.

Proof. We assume that $u_m = o(d_m)$ is a Tauberian condition for $\mathfrak{S}$.

For each j the sequence $\mathbf{s}^j$ is unbounded because of (6); hence $u_m(j) = o(d_m)$ is not true, and therefore $d_m/|u_m(j)| \to \infty$ is not true. This means (when $m \to \infty$)

$$\liminf d_m/|u_m(j)| < \infty \qquad (j = 0, 1, \ldots).$$

Using (8) we even get

$$\liminf \frac{d_m}{|u_m(j)|} = \liminf \frac{d_m}{|u_m(j+1)|} \left| \frac{u_m(j+1)}{u_m(j)} \right| = 0 \quad (j = 0, 1, \ldots). \tag{12}$$

After this preparation we show that $u_m = O(d_m)$ is a Tauberian condition for $\mathfrak{S}$. Given a sequence $\mathbf{s} \in \mathfrak{S}$ with

$$|u_m| \leqslant Kd_m, \qquad (m = 0, 1, \ldots; \text{ for suitable } K) \tag{13}$$

we have to show that this sequence is convergent. According to (5)

$$u_m = \bar{u}_m + \sum_{j=0}^{\infty} \lambda_j u_m(j) \quad \text{with} \quad \sum_{m=0}^{\infty} \bar{u}_m \quad \text{convergent}, \qquad \lambda_j \to 0.$$

We examine the coefficient λ_0. Because of (6) there is an index m_0 such that $u_m(0) \neq 0$ for $m \geqslant m_0$. Furthermore it follows from (7) that

$$|u_m(j)/u_m(0)| \leqslant 2^{-j}, \qquad (m \geqslant m_0;\ j = 1, 2, \ldots), \tag{14}$$

and from (8) that

$$|u_m(j)/u_m(0)| \to 0, \qquad (m \to \infty; j = 1, 2, \ldots). \tag{15}$$

Because of (14) and $\lambda_j \to 0$ the series on the right-hand side of the equation

$$\frac{u_m}{u_m(0)} = \frac{\bar{u}_m}{u_m(0)} + \lambda_0 + \sum_{j=1}^{\infty} \lambda_j \frac{u_m(j)}{u_m(0)} \qquad (m \geqslant m_0),$$

converges uniformly in m; using (15) together with $\bar{u}_m = o(1)$ and $|u_m(0)| \to \infty$ we get

$$\lim u_m/u_m(0) = \lambda_0 \qquad (\text{as } m \to \infty). \tag{16}$$

From (13) and (12) we derive

$$\liminf |u_m/u_m(0)| = 0.$$

This and (16) yield $\lambda_0 = 0$. In the same manner we show that $\lambda_1 = 0$, $\lambda_2 = 0, \ldots$. It follows that $\mathbf{s} = \bar{\mathbf{s}}$, hence that $\mathbf{s}$ is a convergent sequence.

3. A Corresponding Matrix

We shall construct a regular matrix A whose convergence domain c_A is a sequence space $\mathfrak{S} = \mathfrak{S}(\mathbf{s}^0, \mathbf{s}^1, ...)$ of the form (5). For this purpose we start with sequences $\mathbf{w}^j$ which are linearly independent modulo (m), where (m) is the set of bounded sequences and linear independence refers to finite linear combinations. Using suitable indices $q(j)$ we get the $\mathbf{s}^j$ yielding $\mathfrak{S}$ by formulas (10), (11). The $q(j)$ play an important role; they have to fulfill (4), and additional requirements will be imposed on them recursively (cf. the proof of Lemma 7). The main result of the present section is given in Theorem 8. The sequences $\mathbf{w}^j = \{w_k^{(j)}\}$ described in (9) are linearly independent modulo (m), as demanded in Theorem 8; the corresponding $\mathbf{s}^j$ (defined by (10) or (11) with indices $q(j)$ of type (4)) satisfy the assumptions of Theorem 3. This yields our Theorem 2. Of course we can use many other sets of sequences $\mathbf{w}^j$. In this connection it is useful to observe the following fact: Properties (6) and (8) are not influenced by the choice of the $q(j)$, while property (7) is a consequence of (8) if we make the $q(j)$ increase fast enough.

Lemma 4. *Let C be a regular matrix with*

$$c_{nk} = 0 \quad (k < n), \qquad c_{nn} = 1, \qquad \sum_{k>n} |\, c_{nk} \,| \leqslant \rho < 1. \tag{17}$$

Then the convergence domain c_C consists of all sequences

$$\mathbf{s} = \bar{\mathbf{s}} + \mathring{\mathbf{s}} \qquad \textit{where} \qquad \bar{\mathbf{s}} \in (c) \qquad \textit{and} \qquad C\mathring{\mathbf{s}} = 0. \tag{18}$$

Proof. See [5, Theorem 2, Lemma 2]. We repeat the main features. The matrix $C = I - H$ has a regular inverse $C^{-1} = I + H + H^2 + \cdots$, of type (17) with ρ replaced by $\rho/(1-\rho)$. Hence C maps (c) onto (c). This yields (18).

For later reference we note that under the assumptions of Lemma 4 the boundedness domain m_C consists of all sequences

$$\mathbf{s} = \hat{\mathbf{s}} + \mathring{\mathbf{s}} \qquad \text{where} \qquad \hat{\mathbf{s}} \in (m) \qquad \text{and} \qquad C\mathring{\mathbf{s}} = 0; \tag{19}$$

and that

$$\bar{\mathbf{s}} = C^{-1}(C\mathbf{s}) \qquad \text{for} \quad \mathbf{s} \in c_C\,. \tag{20}$$

Lemma 5. *Let the sequence* $\mathbf{w}$ *be unbounded. Then there exists, for each given* ρ $(0 < \rho < 1)$, *a regular row-finite* B *of type* (17) *whose null space consists of the multiples of* $\mathbf{w}$.

Proof (cf. [5, Theorem 3]). We put $b_{nk} = 0$ for $k < n$ and $b_{nn} = 1$; further each row will contain at most one element $b_{nk} \neq 0$ with $k > n$. To define this element we shall use indices $n_0 < n_1 < \cdots$; for b_{nk} and w_n we shall also write $b(n, k)$ and $w(n)$. We determine n_0 uniquely by demanding

$$w(0) = \cdots = w(n_0 - 1) = 0, \qquad w(n_0) \neq 0 \tag{21}$$

(hence $n_0 = 0$ if $w_0 \neq 0$), and choose $n_1, n_2, \ldots$ recursively such that

$$w(n_i) \neq 0 \qquad (i = 1, 2, \ldots),$$
$$|w(n)/w(n_i)| < \rho/i \qquad \text{for} \quad n_{i-2} < n \leqslant n_{i-1} \qquad (i = 1, 2, \ldots),$$

using the auxiliary index $n_{-1} = n_0 - 1$ (so that in the case $i = 1$ only $n = n_0$ is admitted). For each $n \geqslant n_0$ the equation

$$w(n) + b(n, n_i)\, w(n_i) = 0, \qquad (n_{i-2} < n \leqslant n_{i-1}\,;\; i = 1, 2, \ldots) \tag{22}$$

defines one element $b(n, k)$ with $k > n$. We put $b(n, k) = 0$ for all other $k > n$ (also in the case $n < n_0$, compare (23)). Now B is well defined, regular, row-finite, and of type (17). It follows from (22) that $B\mathbf{w} = 0$. On the other hand each solution of the equation $B\mathbf{x} = 0$ is of the form $\mathbf{x} = \gamma\mathbf{w}$, since it is uniquely determined by the value $x(n_0)$. Hence B has all the desired properties.

We emphasize that (if $n_0 > 0$)

$$b_{nk} = 0 \qquad \text{for} \quad k \neq n, \qquad (n = 0, \ldots, n_0 - 1); \tag{23}$$

this means that the rows $0, 1, \ldots, n_0 - 1$ of B coincide with the corresponding rows of the identity matrix. Further we mention the following fact concerning the equation $B\mathbf{s} = \mathbf{t}$ (also written as $(Bs)_n = t_n$; $n = 0, 1, \ldots$): If $(B\mathbf{s})_n = 0$ for $n = 0, \ldots, n_j$ (with any fixed j), then $s_k = \gamma w_k$ for $k = 0, \ldots, n_j$ (and also for $k = n_{j+1}$) with suitable γ. We shall only need the following part of this assertion: Given an index p there is an index p^* such that

$$(B\mathbf{s})_n = 0 \quad \text{for } n = 0, \ldots, p^* \text{ implies } s_k = \gamma w_k \quad \text{for } k = 0, \ldots, p. \tag{24}$$

Our Lemma 7 and its proof are prepared and illustrated by the following lemma.

LEMMA 6. *Let the sequences* $\mathbf{w}^0, \ldots, \mathbf{w}^r$ *be linearly independent modulo* (m). *Then there exists, for each given* ρ $(0 < \rho < 1)$, *a regular row-finite matrix of type* (17) *whose null space consists of all linear combinations*

$$\mathring{\mathbf{s}} = \gamma_0 \mathbf{w}^0 + \cdots + \gamma_r \mathbf{w}^r.$$

Hence the corresponding convergence domain is given by (18) with $\mathring{\mathbf{s}}$ as above.

Proof (cf. [5, Theorem 3]). We outline the main points. The matrix in question will be given as a product $C(r)$:

$$C(r) = B(r)\,B(r-1)\cdots B(0). \tag{25}$$

Here $B(0)$ is the matrix B of Lemma 5 for $\mathbf{w} = \mathbf{w}^0$; its null space is spanned by $\mathbf{w}^0$. Next $B(1)$ is the B for $\mathbf{w} = B(0)\mathbf{w}^1$ which sequence is unbounded because of (19); we find that the null space of $B(1)\,B(0)$ is spanned by $\mathbf{w}^0$ and $\mathbf{w}^1$. Further $B(2)$ is the B for $\mathbf{w} = B(1)\,B(0)\mathbf{w}^2$, and so on. By making the corresponding bounds $\rho_0, \ldots, \rho_r$ small enough we achieve the bound ρ for $C(r)$.

Considering the equation $C(r)\mathbf{s} = \mathbf{t}$ we prove by successive application of (24): Given any index p there is an index p^* such that

$$(B(r)\cdots B(0)\mathbf{s})_n = 0 \qquad \text{for} \quad n = 0, \ldots, p^*,$$

implies (26)

$$s_k = \gamma_0 s_k^{(0)} + \cdots + \gamma_r s_k^{(r)} \qquad \text{for} \quad k = 0, \ldots, p,$$

with suitable scalars $\gamma_0, \ldots, \gamma_r$.

LEMMA 7. *Let the sequences* $\mathbf{w}^0, \mathbf{w}^1, \ldots$ *be linearly independent modulo* (m) *and let* ρ $(0 < \rho < 1)$ *be given. Then there exist indices* $q(j)$ *of type* (4) *and a regular row-finite* C *of type* (17) *such that the null space of* C *consists of all sequences*

$$\mathring{\mathbf{s}} = \gamma_0\mathbf{s}^0 + \gamma_1\mathbf{s}^1 + \cdots \tag{27}$$

where the $\mathbf{s}^j$ *are defined by* (11) *and the* γ_j *are arbitrary scalars.*

Hence the convergence domain c_C is given by (18) with $\mathring{\mathbf{s}}$ as above. In (27) we use coordinatewise convergence as in (5).

Proof. Using the construction (25) we shall define matrices $B(0), B(1), \ldots$ and indices $q(j), p(j), p^*(j)$ which satisfy

$$q(0) \leqslant p(0) \leqslant p^*(0) < q(1) \leqslant p(1) \leqslant p^*(1) < \cdots$$

and the conditions mentioned below.

We put

$$q(0) = 1 \qquad \text{(which implies that } s_0^{(0)} = 0) \tag{28_0}$$

and choose $p(0) \geqslant q(0)$ such that

$$s_k^{(0)} \neq 0 \qquad \text{for at least one } k \text{ with} \quad q(0) \leqslant k \leqslant p(0). \tag{29_0}$$

$B(0)$ is defined as the matrix B of Lemma 5 with $\mathbf{w} = \mathbf{s}^0$. Using (24) we choose $p^*(0) \geqslant p(0)$ such that

$$(B(0)\mathbf{x})_n = 0 \qquad \text{for} \quad n = 0,\dots, p^*(0),$$

implies $\qquad (30_0)$

$$x_k = \gamma_0 s_k^{(0)} \qquad \text{for} \quad k = 0,\dots, p(0),$$

where γ_0 is uniquely determined because of (29_0).

Next we choose $q(1) > p^*(0)$ such that

$$(B(0)\mathbf{s}^1)_k = 0 \qquad \text{for} \quad k = 0,\dots, p^*(0), \tag{28_1}$$

and $p(1) \geqslant q(1)$ such that

$$s_k^{(1)} \neq 0 \qquad \text{for at least one } k \text{ with} \quad q(1) \leqslant k \leqslant p(1). \tag{29_1}$$

$B(1)$ is defined as the matrix B of Lemma 5 with $\mathbf{w} = B(0)\,\mathbf{s}^1$. Because of$(28_1)$ the rows $0,\dots, p^*(0)$ of $B(1)$ coincide with the corresponding rows of the identity matrix (cf. (23)), hence $B(0)$ and $B(1)\,B(0)$ coincide in the rows $0,\dots, p^*(0)$. Using (26) we choose $p^*(1) \geqslant p(1)$ such that

$$(B(1)\,B(0)\mathbf{x})_n = 0 \qquad \text{for} \quad n = 0,\dots, p^*(1),$$

implies $\qquad (30_1)$

$$x_k = \gamma_0 s_k^{(0)} + \gamma_1 s_k^{(1)} \qquad \text{for} \quad k = 0,\dots, p(1),$$

where γ_0 and γ_1 are uniquely determined because of (29). Further the γ_0 here is the same as in (30_0).

The next step yields $q(2), p(2), p^*(2)$, and $B(2) = B$ with $\mathbf{w} = B(1)\,B(0)\,\mathbf{s}^2$. Similarly as above, $B(2)$ and the identity matrix, and therefore $B(1)\,B(0)$ and $B(2)\,B(1)\,B(0)$ coincide in the rows $0,\dots, p^*(1)$. Furthermore

$$(B(2)\,B(1)\,B(0)\mathbf{x})_n = 0 \qquad \text{for} \quad n = 0,\dots, p^*(2),$$

implies $\qquad (30_2)$

$$x_k = \gamma_0 s_k^{(0)} + \gamma_1 s_k^{(1)} + \gamma_2 s_k^{(2)} \qquad \text{for} \quad k = 0,\dots, p(2),$$

where γ_0, γ_1, γ_2 are uniquely determined, and γ_0, γ_1 are the same as in (30_1).

In this manner we continue. The construction of C consists now in putting together rows $0,\dots, p^*(0)$ of $B(0)$; rows $p^*(0) + 1,\dots, p^*(1)$ of $B(1)\,B(0)$; rows $p^*(1) + 1,\dots, p^*(2)$ of $B(2)\,B(1)\,B(0)$; and so on. Similarly as above, C and $B(j) \cdots B(0)$ coincide in the rows $0,\dots, p^*(j)$. This C is clearly row-finite, and it is of type (17) if we use bounds ρ_0, ρ_1,... with $(1 + \rho_0)(1 + \rho_1) \cdots < 1 + \rho$. Regularity of C, however, is not jet guaranteed. This defect can be repaired by a suitable choice of the $p^*(j)$. We make (for example) $p^*(1)$ not only so large that (30_1) holds, but also so large that

$$\sum_{k=n+1}^{\infty} |(B(1)\,B(0))_{nk}| \leqslant \sum_{k=n+1}^{\infty} (|B(1)|\,|B(0)|)_{nk} \leqslant \rho_2 \qquad \text{for} \quad n > p^*(1).$$

Then the norms of the rows $p^*(1)+1, p^*(1)+2,\ldots$ of $B(1)\,B(0)$ and $B(2)$ are less than $1+\rho_2$, hence the norms of the corresponding rows of $B(2)\,B(1)\,B(0)$ are less than $(1+\rho_2)^2$. Thus we achieve that the norms of the rows $p^*(j)+1,\ldots, p^*(j+1)$ of C are less than $(1+\rho_{j+1})^2$, whence it follows that the row-sums of C are tending to 1.

The proof that all sequences (27) belong to the null space of C is left to the reader. Now let $C\mathbf{x}=0$. This implies, e.g., that $(B(2)\,B(1)\,B(0)\mathbf{x})_k = 0$ for $0 \leqslant k \leqslant p^*(2)$, whence $x_k = \gamma_0 s_k^{(0)} + \cdots + \gamma_2 s_k^{(2)}$ for $0 \leqslant k \leqslant p(2)$ by (30_2). In the same way we show that $x_k = \gamma_0 s_k^{(0)} + \cdots + \gamma_3 s_k^{(3)}$ for $0 \leqslant k \leqslant p(3)$, where $\gamma_0, \gamma_1, \gamma_2$ are the same as before; and so on. It follows that there are uniquely determined constants $\gamma_0, \gamma_1, \ldots$ such that $\mathbf{x}$ coincides with $\gamma_0\mathbf{s}^0 + \gamma_1\mathbf{s}^1 + \cdots$ in any section.

THEOREM 8. *Let the sequences* $\mathbf{w}^0, \mathbf{w}^1,\ldots$ *be linearly independent modulo* (m). *Then there exist indices* $q(j)$ *with* $0 \leqslant q(0) < q(1) < \cdots$ *and a regular row-finite matrix* A *whose convergence domain* c_A *consists of all sequences* $\mathbf{s} = \{s_k\}$ *of the form*

$$\mathbf{s} = \bar{\mathbf{s}} + \sum_{j=0}^{\infty} \lambda_j \mathbf{s}^j \qquad \text{with} \qquad \bar{\mathbf{s}} \in (c) \qquad \text{and} \qquad \lambda_j \to 0, \tag{31}$$

where the $\mathbf{s}^j$ *are defined by*

$$\mathbf{s}^j = \mathbf{w}^j - \{w_0^{(j)},\ldots, w_{q-2}^{(j)}, w_{q-1}^{(j)}, w_{q-1}^{(j)},\ldots\}, \qquad q = q(j). \tag{32}$$

We note that (31), (32) are the same as (5), (11). Alternatively we can construct A such that c_A is given by a modification of (31): No restrictions for the scalars (see Lemma 7), or certain suitable restrictions.

Proof. We start from the C constructed in Lemma 7 and change it with the intention to introduce the restriction that the scalars in (27) form a null sequence. First we explain the basic idea. Let f_n be the row-functionals determined by C and let h_n be functionals with

$$h_n(\mathbf{s}) = \gamma_n, \qquad (\mathbf{s} \in c_C). \tag{33}$$

The sequence of functionals

$$f_0, f_0 + h_0, f_1, f_1 + h_1, \ldots \tag{34}$$

defines a summability method with convergence domain (31): Convergence of the sequence $f_0(\mathbf{s}), f_0(\mathbf{s}) + h_0(\mathbf{s}),\ldots$ implies convergence of $f_n(\mathbf{s})$, hence $\mathbf{s} \in c_C$, and implies also $h_n(\mathbf{s}) \to 0$, hence $\gamma_n \to 0$; and vice versa. In order to get a matrix method we have to replace the h_n by suitable row-functionals (compare [7]).

The convergence domain c_C is an FK-space, see [8, p. 38]. The mapping

$$\mathbf{s} \to \mathring{\mathbf{s}} = \mathbf{s} - \bar{\mathbf{s}} = \mathbf{s} - C^{-1}(C\mathbf{s})$$

(cf. (20)) is linear and continuous. Now γ_0 is determined by $\mathring{s}_k = \gamma_0 s_k^{(0)}$ for the k specified in (29_0). Then $\gamma_1, \gamma_2,\ldots$ are determined recursively by similar formulas. Hence the functionals h_n in (33) exist and are linear and continuous.

Each h_j $(j = 0, 1,\ldots)$ admits a representation

$$h_j(\mathbf{s}) = \sum_{k=0}^{\infty} \alpha_k^{(j)} s_k + \sum_{n=0}^{\infty} \beta_n^{(j)} (C\mathbf{s})_n + \beta_j \lim_{n\to\infty} (C\mathbf{s})_n \tag{35}$$

($\mathbf{s} \in c_C$; cf. [6, p. 476, 5.2]) where

$$\alpha_k^{(j)} = 0 \qquad \text{for} \quad k > k(j), \qquad \sum_{n=0}^{\infty} | \beta_n^{(j)} | < \infty.$$

Inserting in (35) the sequences $\mathbf{s} = \{0,\ldots, 0, 1, 1,\ldots\}$, for which $h_j(\mathbf{s}) = 0$, shows that the β_j are vanishing. Now we approximate $h_j(\mathbf{s})$ by

$$g_j(\mathbf{s}) = \sum_{k=0}^{\infty} \alpha_k^{(j)} s_k + \sum_{n=0}^{m(j)} \beta_n^{(j)} (C\mathbf{s})_n \tag{36}$$

where

$$\sum_{n=m(j)+1}^{\infty} | \beta_n^{(j)} | \to 0 \qquad \text{as} \qquad j \to \infty. \tag{37}$$

Condition (37) assures $g_j(\mathbf{s}) - h_j(\mathbf{s}) \to 0$ as $j \to \infty$ $(\mathbf{s} \in c_C)$. Therefore we can replace the h_j in (34) by the g_j without changing the convergence domain. By rearranging (36) we see that each g_j is a finite row-functional. This concludes the proof.

References

1. G. G. Lorentz, Tauberian theorems and Tauberian conditions, *Trans. Amer. Math. Soc.* **63** (1948), 226–234.
2. G. G. Lorentz and K. Zeller, Series rearrangements and analytic sets, *Acta Math.* **100** (1958), 149–169.

3. G. G. LORENTZ AND K. ZELLER, o- but not O-Tauberian theorems, *Proc. Amer. Math. Soc.* **45** (1974), 401–404.
4. M. STIEGLITZ, Die allgemeine Form der O-Tauber-Bedingung für das Euler-Knopp- und Borel-Verfahren, *J. Reine Angew. Math.* **246** (1971), 172–179.
5. A. WILANSKY AND K. ZELLER, Summation of bounded divergent sequences, topological methods, *Trans. Amer. Math. Soc.* **78** (1955), 501–509.
6. K. ZELLER, Allgemeine Eigenschaften von Limitierungsverfahren, *Math. Z.* **53** (1950/51), 463–487.
7. K. ZELLER, Über den perfekten Teil von Wirkfeldern, *Math. Z.* **64** (1956), 123–130.
8. K. ZELLER UND W. BEEKMANN, "Theorie der Limitierungsverfahren," Springer-Verlag, Berlin, Heidelberg, New York, 1970.